Announcements

1) HW due today, new one up tomorrow or later tonight

Example 1:

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A : \mathbb{C} \to \mathbb{C}^2$$

$$\operatorname{rank}(A) = 1$$

$$\operatorname{ran}(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$$

$= $ the line $y = 2x$.

Use least-squares

to "solve"

$$Ax = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ not in } \operatorname{ran}(A) \right)$

Let $\tilde{A} = \begin{bmatrix} 1.0001 \\ 2 \end{bmatrix}$

$$\tilde{b} = \begin{bmatrix} .9999 \\ 0 \end{bmatrix}$$

We should expect the number $K(A) = \|A\|_2 \|A^+\|_2$ to show up since that's what appeared when solving $Ax = b$ with $A \in \mathbb{C}^{m \times n}$, invertible

First calculate $K(A)$.

We get $K(A) = 1$.

Let's first consider the problem of changing b to $\tilde{b}$ and then solving for x (and $\tilde{x}$).

We saw that perturbing b to $\tilde{b}$ yielded relative errors less than the condition number of A when solving for either x or $y = Ax$.

Now we perturb A to $\tilde{A}$ and investigate what happens to solutions x or $y = Ax$.

We calculate

$$\frac{|x-\tilde{x}|}{|x|} \cdot \frac{\|A\|_2}{\|A-\tilde{A}\|_2}.$$

We want this number to be less than or equal to

$K(A) = 1.$

Unfortunately,

we get

$$\frac{|x-\tilde{x}|}{|x|} \cdot \frac{\|A\|_2}{\|A-\tilde{A}\|_2}$$

$$\approx 1.341542 > 1.$$

So there is more to the conditioning of this problem than just $K(A)$.

For y, we get

$$\frac{\|y-\tilde{y}\|_2}{\|y\|_2} \cdot \frac{\|A\|_2}{\|A-\tilde{A}\|_2}$$

≈ 2, which

is even worse!

In fact, even b is deceptive. Our matrix was too simple to register any issues.

Theorem: The least-squares problem is well-conditioned, with the following condition numbers:

Perturbing \ Solving for	y	x
b	$\dfrac{1}{\cos\theta}$	$\dfrac{\kappa(A)}{\eta\cos(\theta)}$
A	$\dfrac{\kappa(A)}{\cos(\theta)}$	$\kappa(A) + \dfrac{\kappa(A)^2\tan(\theta)}{\eta}$

Where

$$\kappa(A) = \|A\|_2 \|A^+\|_2 = \frac{\sigma_1(A)}{\sigma_n(A)}$$

$$\cos\theta = \frac{\|y\|_2}{\|b\|_2}$$

$$\eta = \frac{\|A\|_2 \|x\|_2}{\|y\|_2}$$

The numbers in the first row are exact and the numbers in the second row are upper bounds.

Why do these numbers show up?

Perturbing b, solving for x:

$$x = A^+ b$$

$$\tilde{x} = A^+ \tilde{b}.$$

Look at

$$\frac{\|x - \tilde{x}\|_2}{\|x\|_2} \cdot \frac{\|b\|_2}{\|b - \tilde{b}\|_2}$$

$$\frac{\|x-\tilde{x}\|_2}{\|x\|_2} \cdot \frac{\|b\|_2}{\|b-\tilde{b}\|_2}$$

$$= \frac{\|A^+b - A^+\tilde{b}\|_2}{\|A^+b\|_2} \cdot \frac{\|b\|_2}{\|b-\tilde{b}\|_2}$$

$$= \frac{\|A^+(b-\tilde{b})\|_2}{\|A^+b\|_2} \cdot \frac{\|b\|_2}{\|b-\tilde{b}\|_2}$$

$$\leq \frac{\|A^+\|_2 \cancel{\|b-\tilde{b}\|_2}}{\|A^+b\|_2} \cdot \frac{\|b\|_2}{\cancel{\|b-\tilde{b}\|_2}}$$

(definition of $\|A^+\|_2$)

$$= \frac{\|A^+\|_2 \|b\|_2}{\|A^+ b\|_2}$$

$$= \frac{\|A^+\|_2 \|b\|_2}{\|A^+ b\|_2} \cdot \frac{\|y\|_2}{\|y\|_2}$$

$$= \frac{\|b\|_2}{\|y\|_2} \frac{\|A^+\|_2 \|y\|_2}{\|A^+ b\|_2}$$

$$= \frac{1}{\cos\theta} \frac{\|A^+\|_2 \|y\|_2}{\|A^+ b\|_2}$$

$$\frac{1}{\cos\theta} \quad \frac{\|A^+\|_2 \|y\|_2}{\|A^+ b\|_2}$$

$$= \frac{1}{\cos(\theta)} \frac{\|A^+\|_2 \|y\|_2}{\|x\|_2}$$

$$= \frac{1}{\cos(\theta)} \frac{\|A^+\|_2 \|y\|_2}{\|x\|_2} \cdot \frac{\|A\|_2}{\|A\|_2}$$

$$= \frac{1}{\cos(\theta)} \|A^+\|_2 \|A\|_2 \frac{\|y\|_2}{\|A\|_2 \|x\|_2}$$

$$= \frac{1}{\cos(\theta)} K(A) \frac{1}{\eta} = \frac{K(A)}{\eta \cos(\theta)}$$

Equality occurs when

$$\|A^+(b-\tilde{b})\|_2 = \|A^+\|_2 \|b-\tilde{b}\|_2,$$

i.e. $b-\tilde{b}$ is a maximal right singular vector for A^+.

Perturbing b, solving for y:

$$\frac{\|y-\tilde{y}\|_2}{\|y\|_2} \cdot \frac{\|b\|_2}{\|b-\tilde{b}\|_2}$$

$$= \frac{\|b\|_2}{\|y\|_2} \frac{\|y-\tilde{y}\|_2}{\|b-\tilde{b}\|_2}$$

$$= \frac{1}{\cos(\theta)} \frac{\|y-\tilde{y}\|_2}{\|b-\tilde{b}\|_2}$$

If $P = AA^+$, orthogonal projection onto $\operatorname{ran}(A)$, then $y = Pb$, $\tilde{y} = P\tilde{b}$.

So $\|y - \tilde{y}\|_2$

$$= \|Pb - P\tilde{b}\|_2$$

$$= \|P(b - \tilde{b})\|_2$$

$$\leq \|b - \tilde{b}\|_2 .$$

This gives

$$\frac{\|y-\tilde{y}\|_2}{\|y\|_2} \cdot \frac{\|b\|_2}{\|b-b\|_2}$$

$$= \frac{1}{\cos(\theta)} \cdot \frac{\|y-\tilde{y}\|_2}{\|b-\tilde{b}\|_2}$$

$$\leq \frac{1}{\cos(\theta)} \frac{\|b-\tilde{b}\|_2}{\|b-\tilde{b}\|_2}$$

$$= \frac{1}{\cos(\theta)}$$

You get equality when

$b - \tilde{b} \in \operatorname{ran}(A)$

$\Rightarrow \quad P(b - \tilde{b}) = b - \tilde{b}.$